Introduction to Theoretical Ecology Assignment 10

Graphical and Local Stability Analysis of the SIR Model

In the lecture, we have learned the dynamics of the SIR model with demography. Here, we assume that the mortality rate of the infected individuals is the same as that of the susceptible and the recovered individuals (i.e., all δ):

$$\frac{dS}{dt} = \theta - \beta SI - \delta S$$

$$\frac{dI}{dt} = \beta SI - \rho I - \delta I$$

$$\frac{dR}{dt} = \rho I - \delta R$$

Then we have:

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = \theta - \delta N$$

The dynamics of the original SIR system can be understood by analyzing the more tractable two-dimensional SI system:

$$\frac{dS}{dt} = \theta - \beta SI - \delta S$$

$$\frac{dI}{dt} = \beta SI - \rho I - \delta I$$

with $R_{(t)}$ simply being:

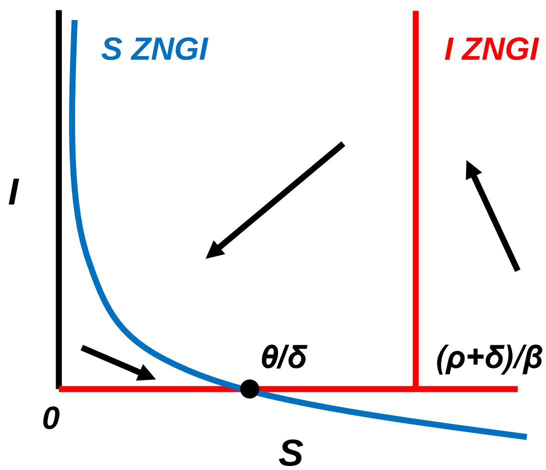
$$R_{(t)} = N_{(t)} - S_{(t)} - I_{(t)}$$

1. Use graphical analysis to determine the stability of the SI system. Please mark the equilibrium points in the phase plane and show the stability criteria. (Hint: There are 2 scenarios; 2.5 pts each)

Solution

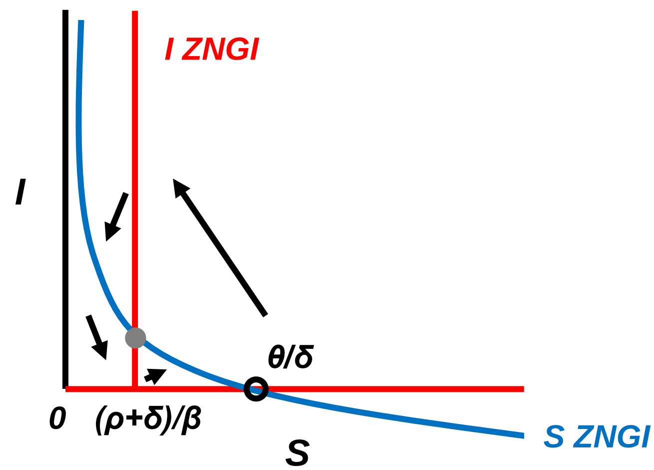
- The ZNGI for *S* is $I = \frac{\theta - \delta S}{\beta S}$
- The ZNGI's for *I* are $I = 0$ and $S = \frac{\rho+\delta}{\beta}$ $\left(= \frac{1}{R_0}\frac{\theta}{\delta}\right)$

Scenario 1: (ρ+δ)/β > θ/δ



In this case (R_0 < 1), the only equilibrium point is (*θ/δ,* 0) and is locally stable.

Scenario 2: (ρ+δ)/β < θ/δ



In this case (R_0 > 1), the first equilibrium point (*θ/δ,* 0) is unstable; the second equilibrium point ((*ρ+δ)/β, θ/(ρ+δ)-δ/β*) is undetermined.

2. Perform local stability analysis on the two equilibrium points and discuss the stability criteria. (2.5 pts for each equilibrium point)

Solution

(1) Disease-free equilibrium (S^*, I^*) = $(\frac{\theta}{\delta}, 0)$:

- Jacobian: $\begin{vmatrix} -\delta & -\beta\frac{\theta}{\delta} \\ 0 & \beta\frac{\theta}{\delta} - \rho - \delta \end{vmatrix}$
- Eigenvalues: $-\delta$ and $\beta\frac{\theta}{\delta} - \rho - \delta$
- Stability criteria: stable if $\beta\frac{\theta}{\delta} - \rho - \delta < 0$, or $\frac{\beta}{\rho+\delta}\frac{\theta}{\delta} < 1$(i.e., $R_0 < 1$)

(2) Endemic equilibrium (S^*, I^*) = $(\frac{\rho+\delta}{\beta}, \frac{\theta}{\rho+\delta} - \frac{\delta}{\beta})$:

- Jacobian: $\begin{vmatrix} -\beta I^* - \delta & -\beta S^* \\ \beta I^* & 0 \end{vmatrix}$
- Characteristic equation: $\lambda^2 + (\beta I^* + \delta)\lambda + \beta^2 S^* I^* = 0$
- Stability criteria: $\lambda_1 + \lambda_2 = -(\beta I^* + \delta) < 0$ and $\lambda_1\lambda_2 = \beta^2 S^* I^* > 0$ → stable if $I^* > 0$ (i.e., $R_0 > 1$) (feasibility)